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(Residential Autonomous College under University of Calcutta)

M.A./M.Sc. THIRD SEMESTER EXAMINATION, JANUARY 2015

SECOND YEAR

MATHEMATICS

Date : 10/01/2015

Time : 11 am – 1 pm

Paper : XI

Full Marks : 50

[*Answer <u>any five</u> questions*]

1. a) Define Winding number of a smooth closed curve with respect to a point not on the curve. If $\gamma:[0,1]\to\mathbb{C}$ is a smooth closed curve and $\alpha\notin\gamma$ then prove that the Winding number $n(\gamma;\alpha)$ of γ with respect to α is an integer. [1+4]

   b) Let f be continuous in a simply connected region G and $\int_y f(z)dz=0$ for every closed contour γ lying in G. Prove that f is analytic in G. [5]

2. a) Prove that if an entire function f is bounded on $\mathbb{C}$, then f is a constant function. Hence deduce that if $f(z)=a_0+a_1z+...+a_nz^n$ is a polynomial of degree $n\geq 1$, then the equation f(z) = 0 has atleast one root in $\mathbb{C}$. [3+4]

   b) Let P be a polynomial in $\mathbb{C}$, $P\neq$constant. Prove that $P(\mathbb{C})=\mathbb{C}$. Is the result true for an entire function? Justify your answer. [3]

3. a) Let $\gamma:|z-z_0|=R$ be the circle of convergence of the series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$. Show that the series is uniformly convergent on every compact subset of int γ. [4]

   b) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^z}$ is uniformly convergent in the closed region $\text{Re } z\geq 1+\delta, \delta>0$. [3]

   c) Let $f(z)=\sum_{n=0}^{\infty}z^n$ and $f_1(z)=\frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$. Show that f_1 is a direct analytical continuation of f. [3]

4. a) If f is analytic on the disk $D:|z-\alpha|<R$, then show that at each point $z\in D$, f has a series expansion of the form: $f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n$ where $a_n=\frac{f^n(\alpha)}{n!}, n=0,1,2,...$. [6]

   b) Let f be an entire function such that ∞ is a pole of order p (p > 0), p being a positive integer. Prove that f is a polynomial of degree p. [4]

5. a) If a function f is bounded and analytic in a deleted neighbourhood $0<|z-\alpha|<\delta$ of a point α, then prove that either f is analytic at α or else α is a removable singularity of f. [5]

   b) Show that the function $f(z)=\cosh\left(z+\frac{1}{z}\right)$ has a Laurent expansion in the form :

   $\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_{n=1}^{\infty}a_n(z^n+z^{-n})$ valid in $0<|z|<\infty$, where

   $a_n=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta d\theta, n=0,1,2,...$. [5]

6. a) Let f and g be analytic in a region G such that f(z) = g(z) on a set $S\subset G$ having a limit point $z_0\in G$. Prove that f(z) = g(z) for all $z\in G$. [6]

   b) Prove that given any n points on the unit circle of the complex plane, there is another point z_0 on the same circle such that the product of the distance of the given n points from z_0 is at least one. [4]

7. a) Use Rouché's theorem to find the number of zeros of the polynomial $p(z)=z^6+9z^4+z^3+2z+4$ that lie inside the unit circle. [5]

b) Evaluate **<u>any one</u>** of the following real integrals by contour integration: [5]

i) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

ii) $\int_{0}^{\infty} \frac{\sin x}{x}\,dx$

8. a) Prove that any bilinear transformation can be expressed as a composition of translation, dilation and inversion (some of these may be missing). [5]

b) Find the bilinear tranformation which transforms the points $z=1,0,\infty$ into the points $w=-1,i,-i$ respectively and show that this bilinear transformation maps.

i) $\operatorname{Im} z=0$ on the circle $|w|=1$,

ii) upper half-plane $\operatorname{Im} z>0$ on the disk $|w|<1$,

iii) lower half-plane $\operatorname{Im} z<0$ on the region $|w|>1$. [5]

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